Zoom International Math League
Online Mathematics Competition
Sample Problems
Solutions

The answer to each of these problems is an integer (a positive integer, a negative integer, or 0). There is only one correct answer for each problem.

All answers should be submitted online, in the blank supplied under the question.

The complete contest contains 20 questions, with a time limit of 60 minutes.

Detailed information for the Online ZIML competition can be found online at

http://online-ziml.areteem.org

For registration, please visit

http://event-reg.areteem.org

Division M (Grades up to 8)

M1. The number 2014 can be factored as $2014 = p \times q \times r$, where p, q, r are all prime numbers. What is $p + q + r$?

Solution: 74.

$2014 = 2 \times 19 \times 53$. So $p + q + r = 2 + 19 + 53 = 74$.

M2. One day, Bob rode his bike to school. When school is off, he forgot his bike and walked home instead. He spent a total of 40 minutes on the road for the round trip. If he walked for both directions, he would have spent a total of 55 minutes. How many minutes would he spend for the round trip if he rode his bike for both directions?

Solution: 25.

Biking for one direction saves 15 minutes off 55 minutes, so biking for both directions would save another 15 minutes off the 40 minutes.

M3. In quadrilateral $ABCD$, $\overline{BD}$ is perpendicular to $\overline{AD}$ and $\overline{BC}$. Given that $BD = 8, AB = 17, CD = 10$. Find the area of $ABCD$.

Solution: 84.

$ABCD$ is a trapezoid, whose two bases are $\overline{AD}$ and $\overline{BC}$. Based on Pythagorean Theorem, $AD = 15$ and $BC = 6$, thus $[ABCD] = \dfrac{(15 + 6) \times 8}{2} = 84$.

Division H (Grades 9-12)

H1. Given a positive integer n, define $S(n)$ to be the sum of its digits. For example, $S(2014) = 2+0+1+4 = 7$. How many four digit numbers n have the sum of digits $S(n)$ equal 9?
Solution: 165.
For a 4-digit number, the first digit cannot be 0 and the remaining 3 digits can be 0.
We use 9 sticks, and divide them up into 4 groups using 3 dividers, where each group represents a digit. Since the first digit may not be 0, we reserve one stick for the first group. Then there are 8 remaining sticks and we split it into 4 groups, each of which may be 0. This is a "Stars and Bars" (or "Sticks and Dividers") problem.
Put the 11 objects (8 sticks and 3 dividers) in a row. Among the 11 positions, choose 3 to be dividers, thus the answer is $\binom{11}{3} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165$.

H2. The positive integers are listed in the following table (the first 4 rows are given):

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & 3 & 4 & & & \\
 & & 5 & 6 & 7 & 8 & 9 & & \\
 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

What is the middle number of the 100^{th} row?
Solution: 9901.
The last number in the n^{th} row is n^2. So the first number in the n^{th} row is $(n-1)^2 + 1 = n^2 - 2n + 2$. So the middle number is $\frac{1}{2}(n^2 - 2n + 2 + n^2) = n^2 - n + 1$. Thus the middle number of the 100^{th} row is $100^2 - 100 + 1 = 9901$.

H3. Triangle ABD is a right triangle, and $\angle B = 90°$. Point C is on side $\overline{BD}$, and $BC = 2, AB = 4$. If $\angle CAD = 45°$, find the length of $\overline{CD}$.
Solution: 10.
Extend $\overline{AC}$ to E such that $\angle AED = 90°$. $\triangle ABC \sim \triangle DEC$, thus $DE/CE = AB/BC = 4/2 = 2$. Also $\angle CAD = 45°$, meaning $\triangle AED$ is an isosceles right triangle, so $AE = DE = 2CE$. Since $AC = \sqrt{2^2 + 4^2} = 2\sqrt{5}$, $CE = AC = 2\sqrt{5}$. And the four points A, B, E, D are concyclic, so by Power of a Point, $BC \cdot CD = AC \cdot CE$, thus $2CD = (2\sqrt{5})^2 = 20$. Therefore $CD = 10$.