2020 Online ZIML Convention
Division H
Team Brain Potion Task

Team solutions must be typed and submitted by 3:00pm on August 14th. Be sure to clearly state the names of all team members in the submission.

Any supplemental pictures, graphs, or drawings may be drawn by hand and included with the typed submission. These can either be put in the file, or added as scanned attachments at the end of the document.

Full rules for the Team Brain Potion are available on the convention page.

Symmetry and Introduction to Groups

Before starting the problems below, please watch the recording of the seminar "Symmetry and Groups".

It is available on YouTube with the link: https://youtu.be/fNm2j97D6uM.

Remember a group is a set G that is closed under an operation $\circ$ satisfying the properties:

- For all a, b, and c in G, $a \circ (b \circ c) = (a \circ b) \circ c$.
- There is an identity element e in G with $e \circ a = a \circ e = a$ for all a in G.
- For any a in G there is an inverse a' (often denoted a^{-1}) with $a \circ a' = a' \circ a = e$.

If the operation is clear, the $\circ$ symbol is often omitted. Further, since the operation is associative parentheses can also be omitted. For example,

$$a \circ b = ab \text{ and } a \circ (b \circ c) = abc.$$

Repeated operations are written with exponents. For example,

$$aaa = a^3 \text{ or } abba = ab^2a.$$

Problems start on the next page. There are a total of 8 problems. Most problems have multiple parts.

Review of Symmetries of the Square

Suppose a square has vertices

$$V_1 = (1,1), V_2 = (-1,1), V_3 = (-1,-1), \text{ and } V_4 = (1,-1).$$

It has 8 symmetries, 4 counterclockwise rotations (by $0°$, $90°$, $180°$, and $270°$) and 4 reflections (across the lines $y = 0$, $x = 0$, $y = x$, and $y = -x$). Denote these symmetries

$$R_0, R_{90}, R_{180}, R_{270}, F_{y=0}, F_{x=0}, F_{y=x}, \text{ and } F_{y=-x}.$$

These symmetries form a group under composition of transformations.

1. (Operation Practice)

   (a) For each of the 8 symmetries, explain how the vertices are permuted by the operation. For example, R_0 doesn't change any vertices:

   $$V_1 \mapsto V_1, V_2 \mapsto V_2, V_3 \mapsto V_3, V_4 \mapsto V_4.$$

   (b) Clearly R_0 is the identity and is hence it's own inverse. What are the inverses of the other seven symmetries?

2. (A Group Presentation) It can be slightly tedious to keep track of all the different symmetries when labeled as above. Typically we try to express the members of a group using a few elements. For this group, let

   $$e = R_0 \text{ (the identity)}, r = R_{90}, \text{ and } s = F_{y=x}.$$

   (a) How can we express the four rotations (including R_0!) in terms of r? Remember we can use exponents to simplify written expressions: $r \circ r = rr = r^2$.

   (b) What is the smallest positive integer k such that $r^k = e$? What is the smallest positive integer j such that $s^j = e$? Explain your answers.

   (c) Note the group is not commutative as $rs \neq sr$. In fact we can write $rs = sr^k$ for a positive integer k. What is the smallest such k? Explain your answer.

   (d) Note the answers to the previous two parts imply that any expression with r and s can be simplified to $s^i r^j$ where i is in $\{0, 1\}$ and j is in $\{0, 1, 2, 3\}$. We'll call this the standard form for this group.

   Write the four reflections $F_{y=0}, F_{x=0}, F_{y=x}, F_{y=-x}$ in this form. Explain your answer.

   (e) Simplify the following (that is write them in standard form):

   $$\text{(i) } r^5 s^3 r^7 r^3 s^2, \text{(ii) } ss^2 rrsr^3 s^{34} r^{10}.$$

Symmetries for Regular Polygons

In general we call the group of rotations and reflections for a regular n-gon D_n where $n \geq 3$. This group will always have $2n$ elements, n rotations and n reflections.

Label the n vertices of the n-gon $V_1, V_2, \ldots, V_n$ counterclockwise around the polygon.

3. (D_3: Symmetries of an Equilateral Triangle) Let r be the rotation permuting the vertices

$$V_1 \mapsto V_2, V_2 \mapsto V_3, V_3 \mapsto V_1$$

and s be the reflection permuting the vertices

$$V_1 \mapsto V_1, V_2 \mapsto V_3, V_3 \mapsto V_2.$$

   (a) Write the six elements of D_3 in standard form. Describe how each permutes the vertices. Which are rotations and which are reflections?

   (b) What are the inverses of each element?

   (c) Simplify the following (that is write them in standard form):

$$\text{(i) } rs, \text{(ii) } r^5s^3r^7r^3s^2, \text{(iii) } ss^2rrsr^3s^{34}r^{10}.$$

4. (D_6: Symmetries of an Regular Hexagon) Let r be the rotation permuting the vertices

$$V_1 \mapsto V_2, V_2 \mapsto V_3, V_3 \mapsto V_4, V_4 \mapsto V_5, V_5 \mapsto V_6, V_6 \mapsto V_1$$

and s be the reflection permuting the vertices

$$V_1 \mapsto V_1, V_2 \mapsto V_6, V_3 \mapsto V_5, V_4 \mapsto V_4, V_5 \mapsto V_3, V_6 \mapsto V_2.$$

   (a) Write the twelve elements of D_6 in standard form. Describe how each permutes the vertices. Which are rotations and which are reflections?

   (b) What are the inverses of each element?

   (c) Simplify the following (that is write them in standard form):

$$\text{(i) } rs, \text{(ii) } r^5s^3r^7r^3s^2, \text{(iii) } ss^2rrsr^3s^{34}r^{10}.$$

5. (D_{10}: Symmetries of a Regular Decagon with 10 sides) As in the previous two problems, let r be a rotation permuting the vertices such that $r^{10} = e$ (and 10 is the smallest power such that $r^k = e$) and s be a reflection permuting the vertices. Note we are not saying exactly which reflection or rotation is given because it does not affect any of the questions below.

   (a) Write out the 20 elements of D_{10} in standard form. Which are rotations and reflections?

   (b) What are the inverses of each element?

Group of Permutations: S_n

In the earlier questions about D_4, D_3, and D_6 we saw how the different symmetries permuted the vertices of the polygon. For example, a symmetry could permute the vertices V_1, V_2, V_3, V_4 of a square as:

$$V_1 \mapsto V_3, V_2 \mapsto V_4, V_3 \mapsto V_1, V_4 \mapsto V_2.$$

Note this naturally corresponds to the permutation of $(1, 2, 3, 4)$ sending

$$1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 2.$$

As a permutation we'll write this as $(3, 4, 1, 2)$.

Not all permutations of $(1, 2, 3, 4)$ correspond to symmetries of a square. For example the permutation, $(2, 1, 3, 4)$, that is,

$$1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4$$

does not correspond to any geometric symmetries of a square.

The permutations of $(1, 2, \ldots, n)$ form a group, called S_n, where the permutations are performed one after another. For example, in S_4,

$$(3, 4, 1, 2) \circ (2, 1, 3, 4) = (3, 4, 2, 1)$$

because

$$1 \mapsto 3 \mapsto 3, 2 \mapsto 4 \mapsto 4, 3 \mapsto 1 \mapsto 2, 4 \mapsto 2 \mapsto 1.$$

In all groups of permutations, the permutation $(1, 2, \ldots, n)$, that is,

$$1 \mapsto 1, 2 \mapsto 2, \ldots, n \mapsto n,$$

is the identity of S_n.

6. (Comparing D_n and S_n) For any $n \geq 3$, a symmetry in D_n can be thought of as a permutation in S_n as described above.

   (a) In general D_n has $2n$ elements. How many elements are in S_n?

   (b) Above we explained how not all the permutations in S_4 correspond to symmetries in D_4. Is it ever the case that all the permutations in S_n correspond to symmetries in D_n? Explain your answer. How does part (a) help justify your answer?

7. (Calculation Practice with S_4) For this problem we'll be working with elements in S_4, permutations of $(1,2,3,4)$.

   (a) What is $(4,3,2,1) \circ (2,3,1,4)$? Explain your answer.

   (b) What is the inverse of $(1,3,2,4)$? Explain your answer.

   (c) Calculate
   $$(2,3,4,1)^2, (2,3,4,1)^3, \ldots$$
   What is the smallest positive integer k such that $(2,3,4,1)^k = e$?

8. (Calculation Practice with S_6)

   (a) What is $(6,5,4,3,2,1) \circ (2,3,4,1,5,6)$? Explain your answer.

   (b) What is the inverse of $(1,5,2,3,4,6)$? Explain your answer.

   (c) Calculate
   $$(2,3,4,5,6,1)^2, (2,3,4,5,6,1)^3, \ldots$$
   What is the smallest positive integer k such that $(2,3,4,5,6,1)^k = e$?

   (d) Calculate
   $$(2,3,1,5,4,6)^2, (2,3,1,5,4,6)^3, \ldots$$
   What is the smallest positive integer k such that $(2,3,1,5,4,6)^k = e$?