2020 Online ZIML Convention
Division M
Team Brain Potion Task

Team solutions must be typed and submitted by 3:00pm on August 14th. Be sure to clearly state the names of all team members in the submission.

Any supplemental pictures, graphs, or drawings may be drawn by hand and included with the typed submission. These can either be put in the file, or added as scanned attachments at the end of the document.

Full rules for the Team Brain Potion are available on the convention page.

Taxicab Geometry and Metric Spaces

Before starting the problems below, please watch the recording of the seminar "Taxicab Geometry and Intro to Metrics".

It is available on YouTube with the link: https://youtu.be/xMnVnSpgf6s.

Remember a metric is a way to measure distances between two points. It must satisfy the following 4 properties for points P and Q:

- $d(P, Q) \geq 0$
- $d(P, Q) = 0$ if and only if $P = Q$
- $d(P, Q) = d(Q, P)$
- $d(P, Q) \leq d(P, R) + d(R, Q)$ for any point R

Problems start on the next page. There are a total of 8 problems. Most problems have multiple parts.

Review of Euclidean and Taxicab Distances

Given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

- The Euclidean distance between P and Q is
$$d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
- The Taxicab distance between P and Q is
$$d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|.$$

1. (Distance Practice) Let $P = (1,1)$, $Q = (4,5)$, and $R = (-8,0)$.
   (a) Calculate $d_E(P,Q)$ and $d_T(P,Q)$.
   (b) Calculate $d_E(Q,R)$ and $d_T(Q,R)$.
2. (Comparing d_E and d_T)
   (a) Give an example of points P, Q, R and S with $d_E(P,Q) = d_E(R,S)$ but $d_T(P,Q) \neq d_T(R,S)$.
   (b) Give an example of points P, Q, R and S with $d_T(P,Q) = d_T(R,S)$ but $d_E(P,Q) \neq d_E(R,S)$.
   (c) Give an example of points P and Q with $d_E(P,Q) < d_T(P,Q)$.
   (d) When is it true that $d_E(P,Q) = d_T(P,Q)$ for points P and Q? Explain your answer.

Circles with Euclidean and Taxicab Distances

Recall a circle is the collection of points a fixed distance (called the radius) from a point (called the center).

Thus a circle will look different if we measure distance with d_E or with d_T.

Given a radius r and center C we define:

- Euclidean Circle: The collection of points P such that $d_E(P, C) = r$.
- Taxicab Circle: The collection of points P such that $d_T(P, C) = r$.

3. (Graphing Circles)

   (a) Graph the Euclidean and Taxicab Circles with center $(2, 0)$ and radius 3 on the same graph. Find all the points where the two circles intersect. How many intersection points are there?

   (b) Graph the Euclidean and Taxicab Circles with center $(-2, 3)$ and radius 5 on the same graph. Find all the points the two circles intersect. How many intersection points are there?

   (c) Explain why the number of intersection points (as in the above two problems) is always the same.

"Perpendicular Bisectors" with Euclidean and Taxicab Distances

Recall with the Euclidean distance the perpendicular bisector of points A and B gives the collection of points that are equal distances from A and B.

Given two points A and B we define:

- Euclidean PB between A and B: The collection of points P such that $d_E(P, A) = d_E(P, B)$.
- Taxicab PB between A and B: The collection of points P such that $d_T(P, A) = d_T(P, B)$.

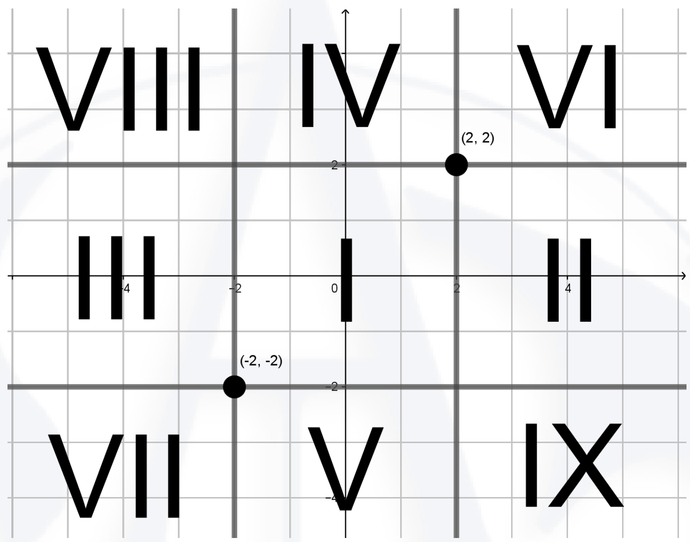
4. (Graphing PBs)

   (a) Graph the Euclidean and Taxicab PBs between points $(-2, 0)$ and $(2, 2)$ on the same graph. Find all points where the PBs intersect.

   (b) Graph the Euclidean and Taxicab PBs between points $(0, -2)$ and $(2, 2)$ on the same graph. Find all points where the PBs intersect.

   (c) Given two points, explain how to find one point that will always be on both the Euclidean and Taxicab PBs.

5. Taxicab PBs can be more complicated than Euclidean PBs. In this problem we consider the Taxicab PB between $A = (-2, -2)$ and $B = (2, 2)$. It will be helpful to divide the coordinate plane into 9 regions (labeled with Roman Numerals):



   (a) First consider region I. What does the Taxicab PB look like in this region? You will graph this at the end of the problem, for now you can just describe it.

   (b) Consider regions II, III, IV, and V next. Fill in the blanks in each of the following statements with points A or B. Explain your answers. Here closer means the Taxicab distance is smaller.

   - All the points in region II are closer to point __ than they are to point __.
   - All the points in region III are closer to point __ than they are to point __.
   - All the points in region IV are closer to point __ than they are to point __.
   - All the points in region V are closer to point __ than they are to point __.

   (c) You're on your own for regions VI, VII, VIII, and IX. What happens in these regions? Explain your answers. Remember you can always try out some points in each region to see if you can find a pattern.

   (d) Put it all together! Graph and describe all the points on the Taxicab PB between $(-2, -2)$ and $(2, 2)$.

Distances with Binary Bytes

In computer science, a byte is a 8-digit sequence of 0s and 1s. For example 00110101 and 10001000 are both bytes. There are $2^8 = 256$ total bytes:

$$00000000, 00000001, 00000010, \ldots, 11111111.$$

We'll consider bytes our points for the problems in this section.

Define the Hamming distance d_H between two bytes P and Q as

$$d_H(P, Q) = \text{ the number of times the digits in } P \text{ and } Q \text{ are different.}$$

For example, if

$$P = 00110011$$
$$\text{and } Q = 00010001$$

then $d_H(P, Q) = 2$ because the third and seventh digits are different.

6. (Hamming Distance Practice)
   (a) Calculate $d_H(P, Q)$ if $P = 01010101$ and $Q = 11101110$.
   (b) Calculate $d_H(P, Q)$ if $P = 11010100$ and $Q = 11000001$.
   (c) Calculate $d_H(P, Q)$ if $P = 01010101$ and $Q = 10101010$.
   (d) What is the maximum Hamming distance between to bytes? Explain your answer.
7. (Hamming Circle) Given a radius r and center C we define a Hamming Circle as the collection of points P such that $d_H(P, C) = r$.
   (a) Describe the Hamming Circle with radius 8 and center 00001111. List all the points on this circle. How many points are there?
   (b) Describe the Hamming Circle with radius 2 and center 00001111. List at least 5 points on this circle. How many points are there on the circle in total?
8. (Hamming PB) Given two points A and B we define a Hamming PB between A and B as the collection of points P such that $d_H(P, A) = d_H(P, B)$.
   (a) Consider the Hamming PB between $A = 01010101$ and $B = 10101010$. List at least 4 points on this PB. In fact, all the points on this PB are all the same distance from A. Explain why this is true.
   (b) Consider the Hamming PB between $A = 01010101$ and $B = 11101110$. Not all the points on this PB are the same distance from A. What possible distances are there? Given an example for each and explain your answer.
   (c) Consider the Hamming PB between $A = 11010100$ and $B = 11000001$. How many points are on the PB? Explain your answer.